Completely Reachable Automata

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Given a DFA $\mathscr{A} = \langle Q, \Sigma \rangle$, a non-empty subset $P \subseteq Q$ is reachable in $\mathscr{A}$ if $P = Q.w$ for some word $w \in \Sigma^*$. A DFA is completely reachable if every non-empty set of its states is reachable.

A DFA $\mathscr{A} = \langle Q, \Sigma \rangle$ is synchronizing if there are a word $w \in \Sigma^*$ and a state $f \in Q$ such that the action of w resets $\mathscr{A}$ to f no matter at which state the action started: $q.w = f$ for all $q \in Q$.

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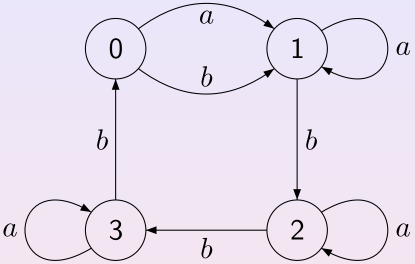

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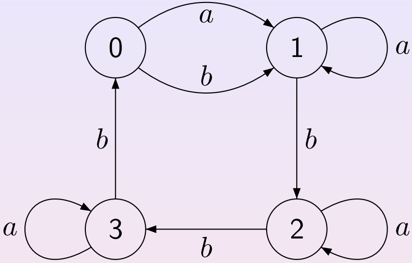

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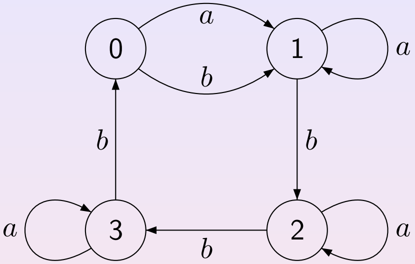

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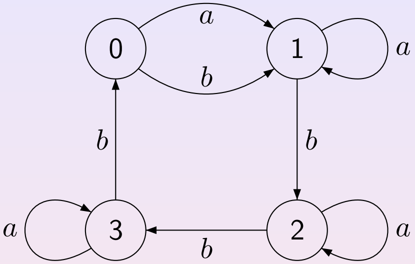

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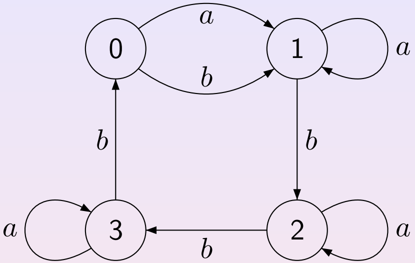

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The states of $\mathscr{C}_n$ are the residues modulo n, and the input letters a and b act as follows:

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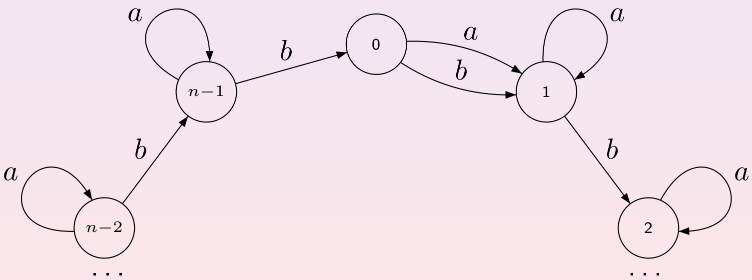

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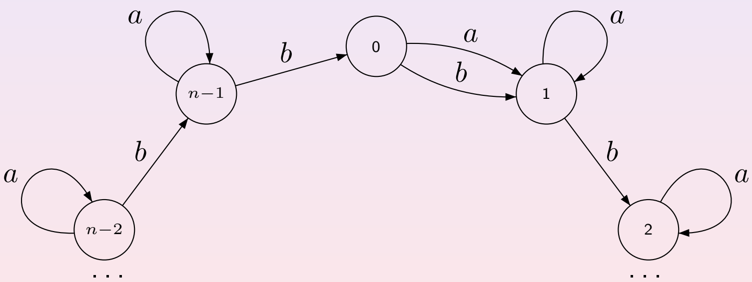

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $n(n-2)+1 = (n-1)^2$.

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In 2019 Yaroslav Shitov found a further improvement to ≈ 0.1654.

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• Avraam Trahtman result for automata whose transition monoid contains no non-trivial subgroups (The Černý conjecture for aperiodic automata, Discrete Math. Theoret. Comp. Sci., 9, no.2 (2007), 3–10).

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For an illustration, consider the power-set automaton of the Černý automaton $\mathscr{C}_4$.

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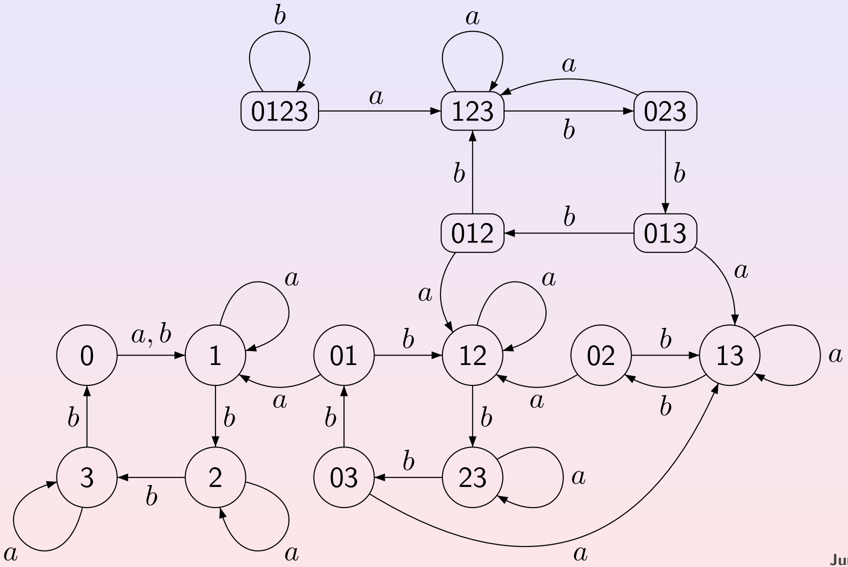

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There are further reasons to study complete reachability . . . which we skip since we need time to introduce certain notions that are used for our main results.

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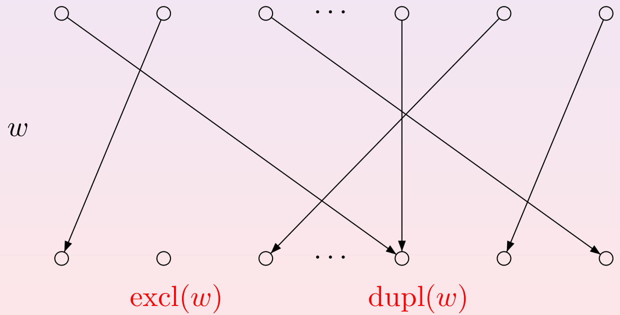

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Theorem (Bondar and MV, DCFS 2016)

If a DFA $\mathscr{A} = \langle Q, \Sigma \rangle$ is such that the graph $\Gamma_1(\mathscr{A})$ is strongly connected, then $\mathscr{A}$ is completely reachable; more precisely, for every non-empty subset $P \subseteq Q$, there is a product w of words of defect 1 such that $P = Q.w$.

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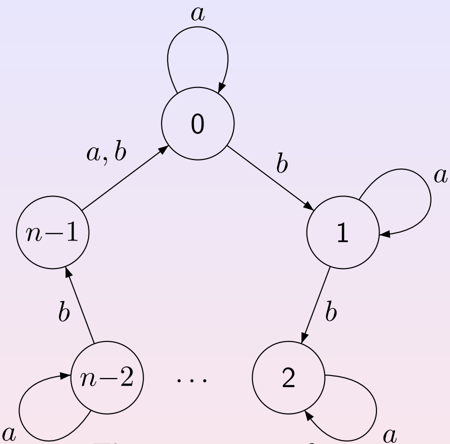

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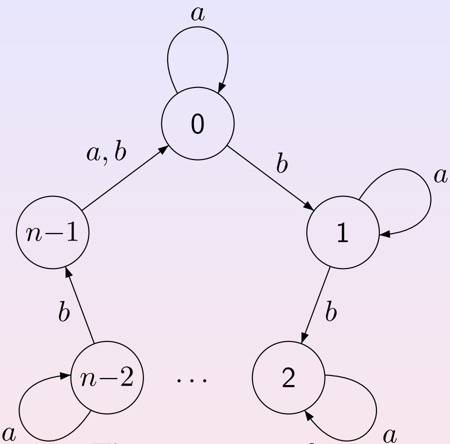

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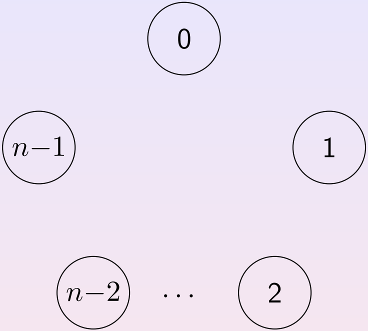

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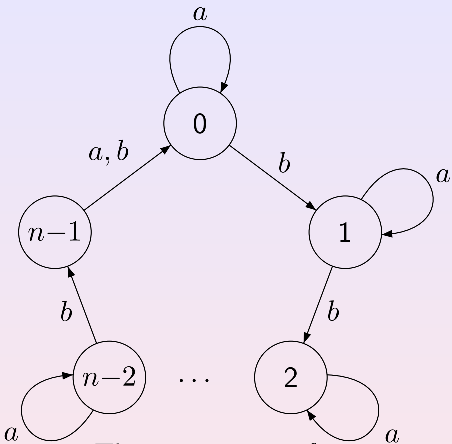

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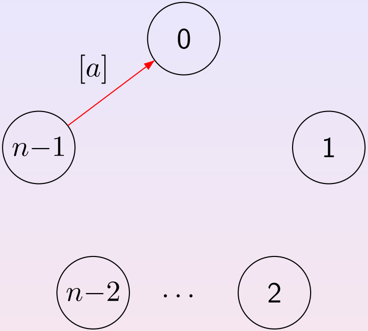

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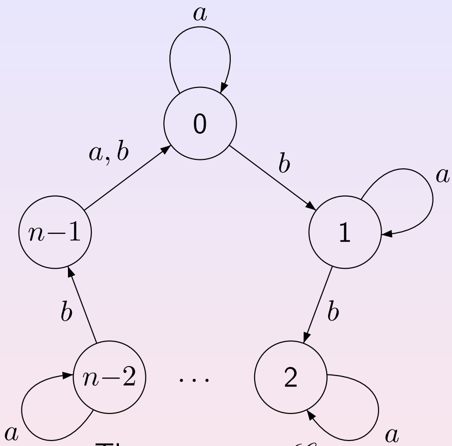

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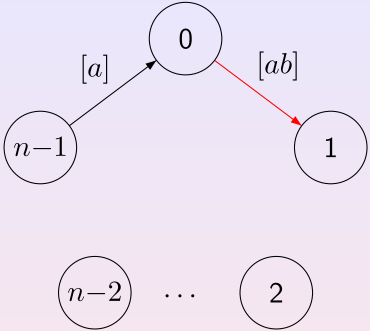

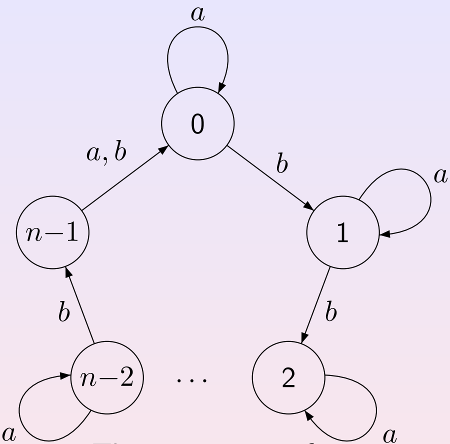

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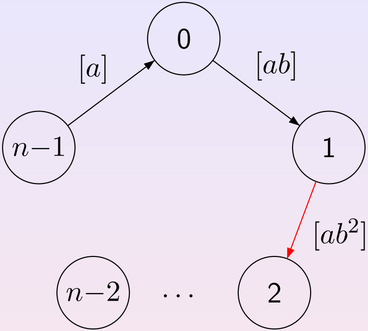
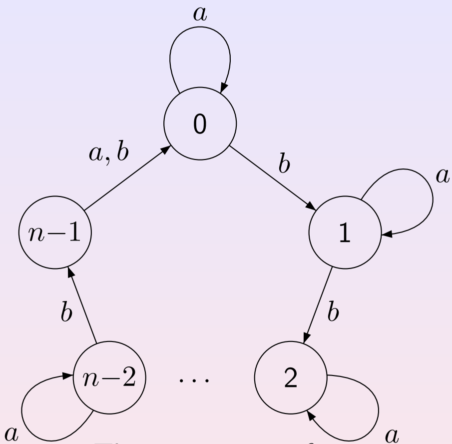


Example



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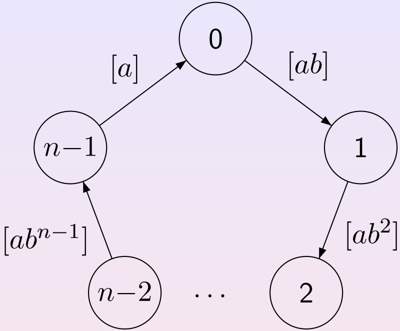

$\Gamma_1(\mathscr{C}_n)$ is strongly connected

The converse of this theorem does not hold: if $\mathscr{A}$ is a completely reachable automaton, and even if for every non-empty subset $P \subseteq Q$, there is a product w of words of defect 1 such that $P = Q.w$, the graph $\Gamma_1(\mathscr{A})$ need not be strongly connected.

Graphs $\Gamma_k(\mathscr{A})$

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Constructing $\Gamma_k(\mathscr{A})$

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Otherwise, we let $R_k := R_{k-1} \cup Q_k$ where
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Now $\Gamma_k(\mathscr{A}) := \big(R_k, J_k\big)$.

Example

Consider the DFA $\mathscr{E}_5$ with 5 states $1, 2, 3, 4, 5$ and 8 input letters $a_{[1]}, a_{[2]}, a_{[3]}, a_{[4]}, a_{[5]}, a_{[1,2]}, a_{[4,5]}, a_{[1,3]}$ whose actions are shown in the following table:

	$a_{[1]}$	$a_{[2]}$	$a_{[3]}$	$a_{[4]}$	$a_{[5]}$	$a_{[1,2]}$	$a_{[4,5]}$	$a_{[1,3]}$
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2	2	1	1	2	2	3	1	4
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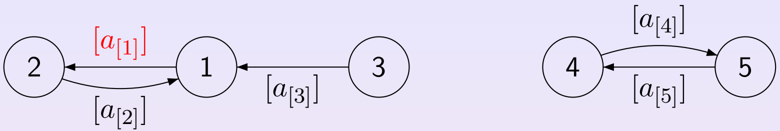
5	5	4	5	5	4	5	3	5
defect	1	1	1	1	1	2	2	3

.

Graph $\Gamma_1(\mathscr{E}_5)$



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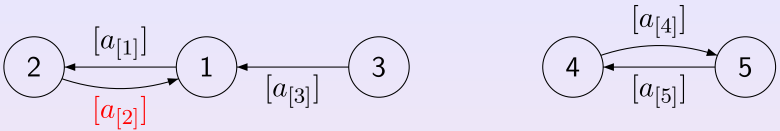

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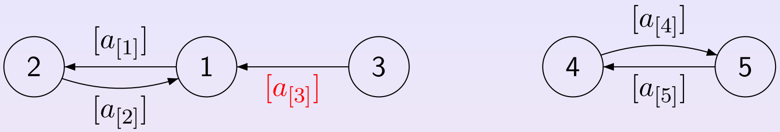

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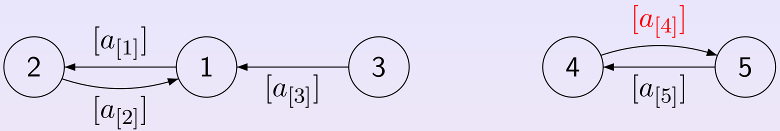

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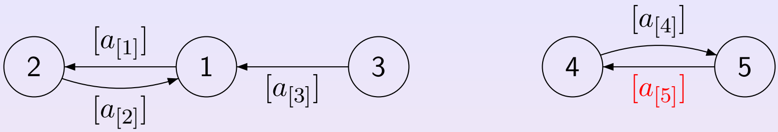

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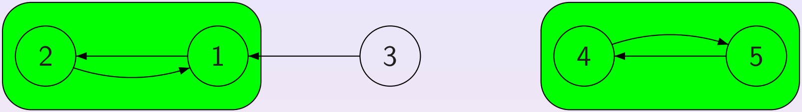
Graph $\Gamma_2(\mathscr{E}_5)$

SCCs of $\Gamma_1(\mathscr{E}_5)$



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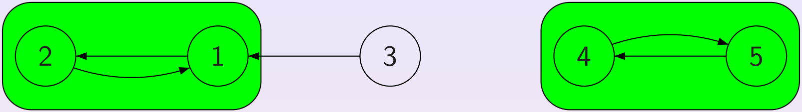

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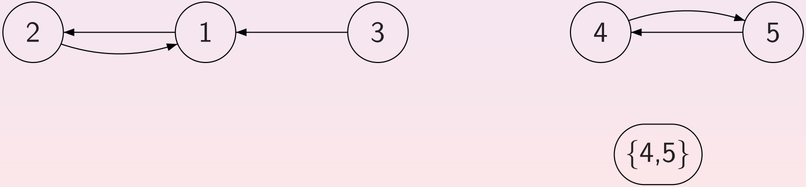

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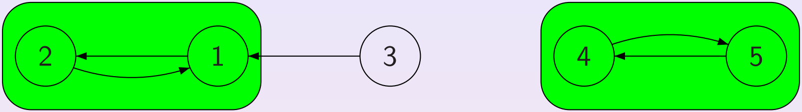

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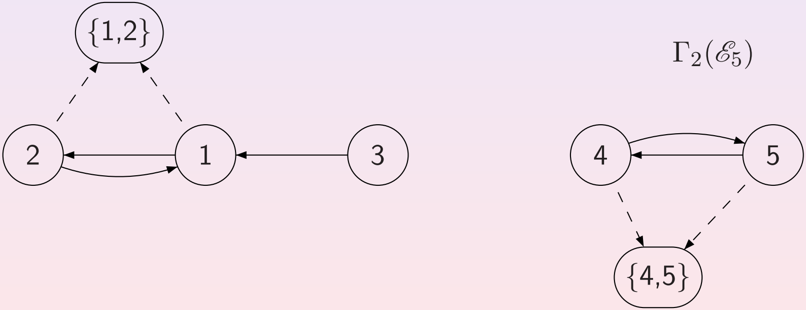

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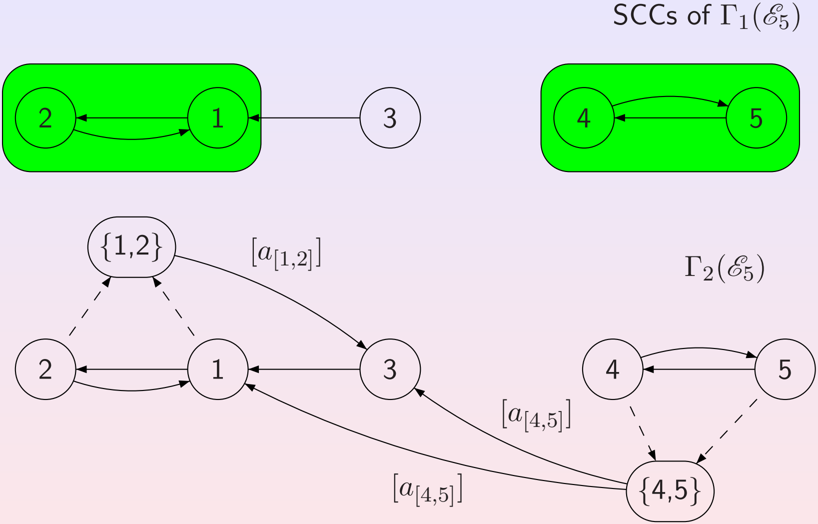

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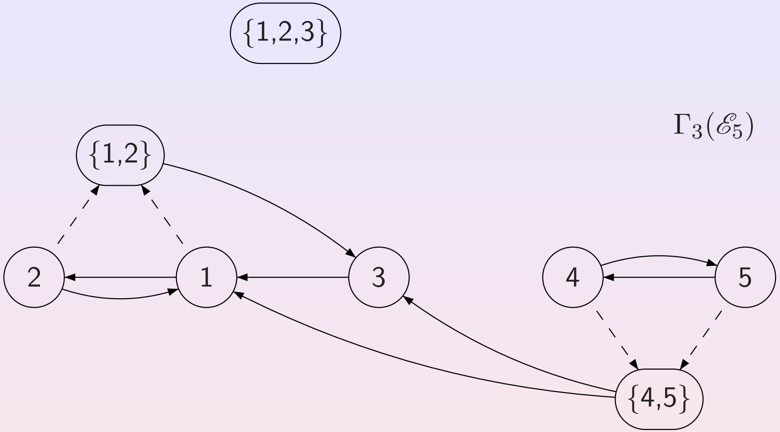

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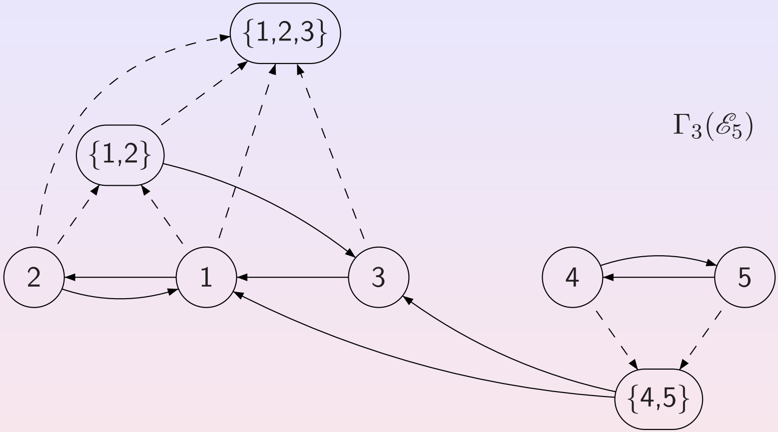

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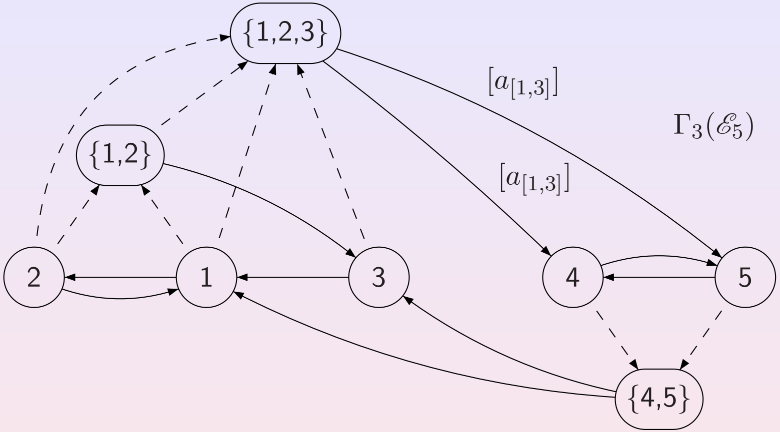

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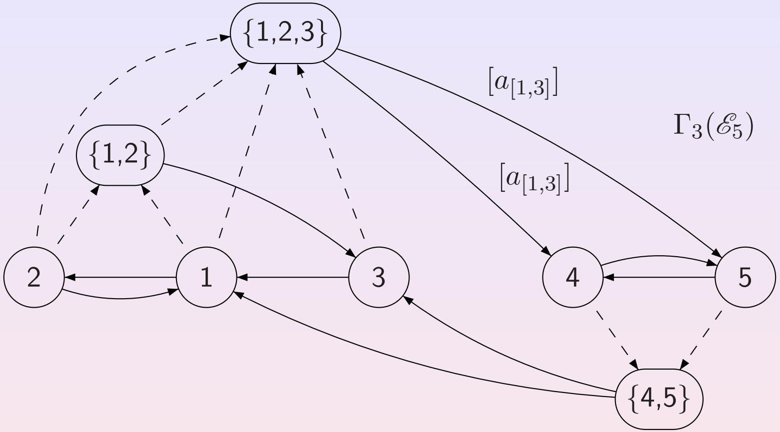

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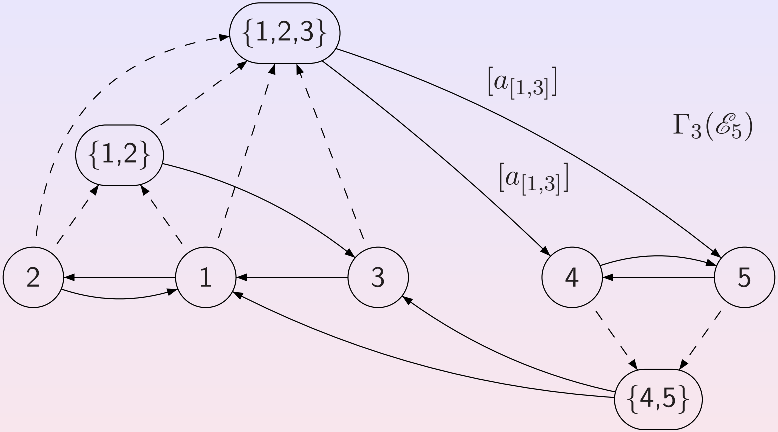

We see that the graph $\Gamma_3(\mathscr{E}_5)$ is strongly connected, whence our process applied to $\mathscr{E}_5$ stops with SUCCESS.

Clearly, for a DFA $\mathscr{A}$ with n states, constructing the sequence of graphs $\Gamma_1(\mathscr{A})$, $\Gamma_2(\mathscr{A})$, ... must stop after at most $n-1$ steps.

Main Results

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Theorem 1

If for DFA $\mathscr{A} = (Q, \Sigma)$, the described process stops at step k with SUCCESS (i.e., the graph $\Gamma_k(\mathscr{A})$ is strongly connected), then $\mathscr{A}$ is completely reachable; more precisely, for every non-empty subset $P \subseteq Q$, there is a product w of words of defect at most k such that $P = Q.w$.

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Theorem 2

If for DFA $\mathscr{A} = (Q, \Sigma)$, the described process stops at step k with FAILURE, then $\mathscr{A}$ is not completely reachable; more precisely, some subset in Q with at least $|Q| - k$ states is not reachable in $\mathscr{A}$.

Characterization

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Theorem 3

A DFA $\mathscr{A}$ is completely reachable if and only if the graph $\Gamma(\mathscr{A})$ is strongly connected.

Complexity

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François Gonze and Raphaël Jungers (DLT 2018) developed a polynomial algorithm for constructing the graph $\Gamma_1(\mathscr{A})$

Complexity, continued

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The following polynomiality conjecture is a weaker version of a conjecture suggested by Henk Don: there exists a constant c such that in every completely reachable automaton with n states, each non-empty subset can be reached by a word of length n^c.

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The following polynomiality conjecture is a weaker version of a conjecture suggested by Henk Don: there exists a constant c such that in every completely reachable automaton with n states, each non-empty subset can be reached by a word of length n^c. If this conjecture holds, then Theorem 3 implies that the problem of whether a given DFA is completely reachable lies in NP.

Complexity, continued

David Casas has managed to extend the approach of Gonze and Jungers to master a polynomial algorithm for constructing the graph $\Gamma(\mathscr{A})$ provided that all letters of $\mathscr{A}$ have defect at most 1.

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It is known that there is no constant C such that in every DFA with n states (not necessarily completely reachable!), each reachable subset can be reached by a word of length n^C.

Synchronization

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$\frac{7}{48} \approx 0.1458333$ improves on the best bound known for general synchronizing automata (with the leading coefficient ≈ 0.1654). Still, we fell short to get a quadratic upper bound so far.

Happy Birthday Werner!!

Wir gratulieren herzlich zum Geburtstag und wünschen alles Gute!