The validity of weighted automata

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Joint work with

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CNRS / Université de Bordeaux / Institut Polytechnique de Bordeaux

Recent Advances of Quantitative Models in Computer Science

22 June 2021

Dedicated to

Professor Werner Kuich

on the occasion of his 80th birthday

with a special thought for his long term collaborator, our friend

Zoltan Ésik

(1951 – 2016)

The results presented in this talk are based
on a work that has been published in the

International Journal of Algebra and Computation **23** (2013)

This work addresses, and proposes a solution to,

the problem of ε-transition removal in weighted automata.

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The solution is based on a new, and more constrained,
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The definition insures that
algorithms are successful on valid automata.

In some (interesting) cases, we are able to establish that
the success of algorithms implies the validity of automata.

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This solution provides a sound theoretical framework for

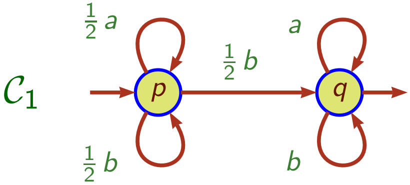
the algorithms implemented in AWALI.

(Downloadable at `vaucanson-project.org/Awali`.)

Program

- The model of weighted automata
- The ε-transition removal for automata
- The problem of ε-transition removal for weighted automata
- Definition of behaviour of weighted automata
- Definition of validity of weighted automata
- Two cases for which validity is decidable
- Conclusion
- The many hidden parts

The weighted automaton model



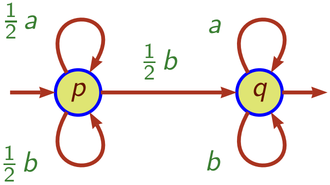
Automata are labelled graphs

Labels are monomials: letters with coefficients

Automata realise functions that map words to values

The weighted automaton model

$\mathcal{C}_1$



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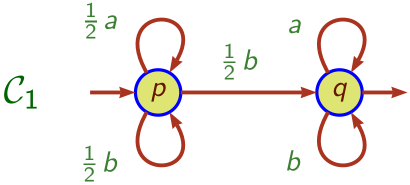

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- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$$b\,a\,b \quad \longmapsto \quad \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

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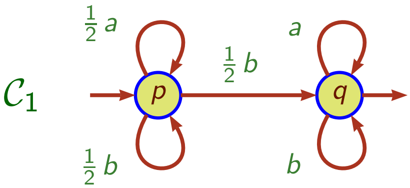

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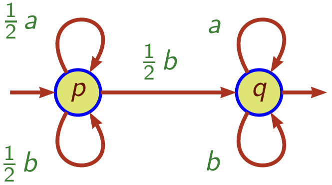

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$$|\mathcal{C}_1| \colon A^* \to \mathbb{Q}$$

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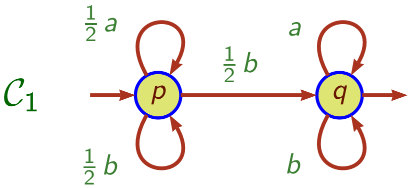

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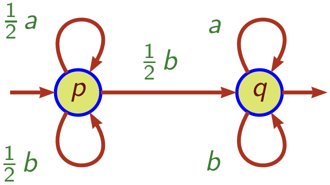

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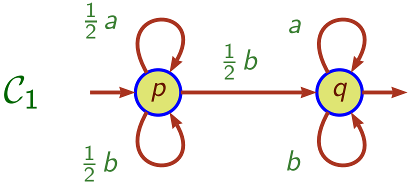
$$|\mathcal{C}_1| = \frac{1}{2}\,b + \frac{1}{4}\,a\,b + \frac{1}{2}\,b\,a + \frac{3}{4}\,b\,b + \frac{1}{8}\,a\,a\,b + \frac{1}{4}\,a\,b\,a + \frac{3}{8}\,a\,b\,b + \frac{1}{2}\,b\,a\,a + \ldots$$

The weighted automaton model



$$\mathcal{C}_1 = \langle\, I_1, \underline{E_1}, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\,a + \frac{1}{2}\,b & \frac{1}{2}\,b \\ 0 & a+b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

The weighted automaton model

$$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle \qquad\qquad \underline{E} = \text{adjacency matrix}$$

$$\begin{aligned} \underline{E}_{p,q} &= \sum \{\mathbf{wl}(e) \mid e \quad \text{transition from } p \text{ to } q\} \\ &= \qquad\qquad \text{linear combination of letters in } A \end{aligned}$$

$$\underline{E}^{n}_{p,q} = \sum \{\mathbf{wl}(c) \mid c \quad \text{computation from } p \text{ to } q \text{ of length } n\}$$

$$\underline{E}^{*} = \sum_{n \in \mathbb{N}} \underline{E}^{n}$$

Since $\underline{E}$ is proper, $\underline{E}^{*}$ is well-defined

$$\underline{E}^{*}_{p,q} = \sum \{\mathbf{wl}(c) \mid c \quad \text{computation from } p \text{ to } q\}$$

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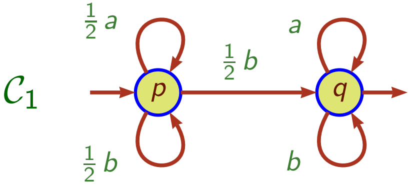

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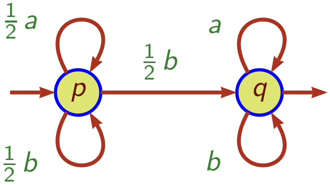

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Every $\mathbb{K}$-automaton defines a series in $\mathbb{K}\langle\langle A^* \rangle\rangle$

whose coefficients are effectively computable

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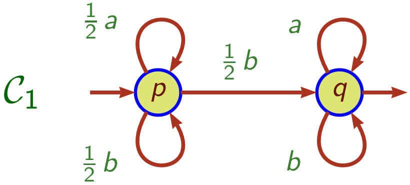

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Where is the problem ?

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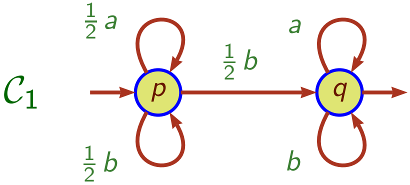

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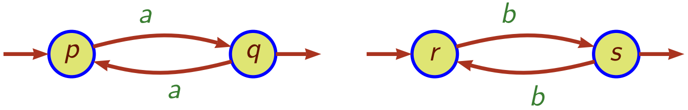
Where is the problem ?

We want to be able to deal with weighted automata where transitions *might be* labelled by the empty word

The need for a richer model: eg, the concatenation product



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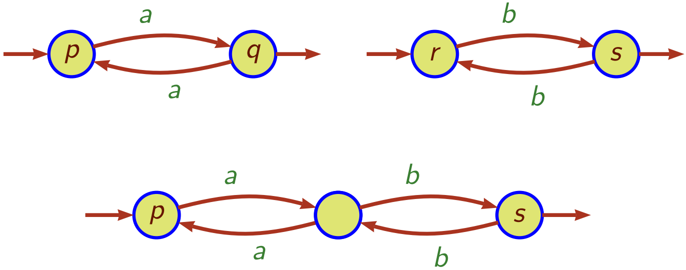

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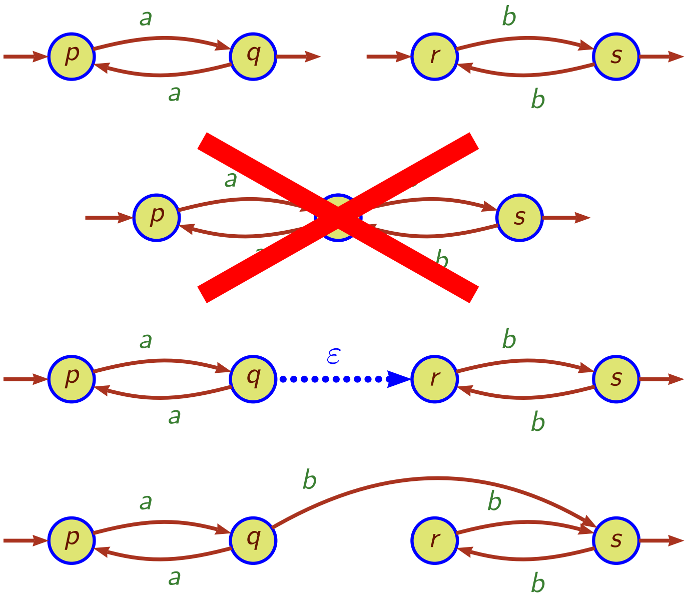

A basic result in (classical) automata theory

Theorem (Folk–Lore)

Every ε-NFA is equivalent to an NFA

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Usefulness of ε-transitions:

Preliminary step for many constructions on NFA's:

- *Product* and *star* of position (Glushkov, standard) automata
- *Thompson construction*
- Construction of the *universal automaton*
- Computation of the *image of a transducer*
- ...

May correspond to the *structure* of the computations

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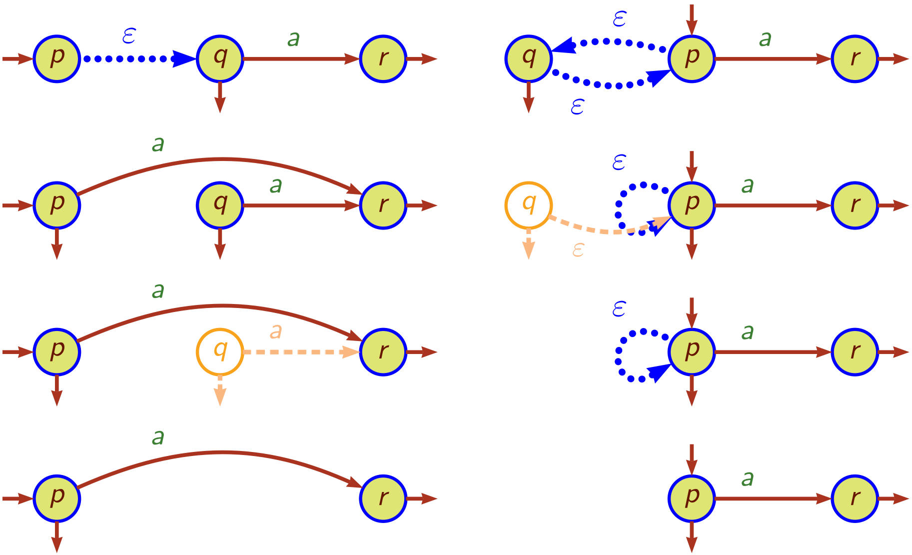
Removal of ε-transitions is implemented in all automata software

A basic result in (classical) automata theory



A basic result in (classical) automata theory

Theorem (Folk–Lore)

Every ε-NFA is equivalent to an NFA

A proof

$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle$ $\qquad$ $\underline{E}$ transition matrix of $\mathcal{A}$

Entries of $\underline{E}$ = subsets of $A \cup \{\varepsilon\}$

$$L(\mathcal{A}) = I \cdot \underline{E}^* \cdot T$$

$$\underline{E} = \underline{E}_0 + \underline{E}_{\mathsf{p}}$$

$$L(\mathcal{A}) = I \cdot (\underline{E}_0 + \underline{E}_{\mathsf{p}})^* \cdot T = I \cdot (\underline{E}_0^* \cdot \underline{E}_{\mathsf{p}})^* \cdot \underline{E}_0^* \cdot T$$

$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle$ equivalent to $\mathcal{B} = \left\langle\, I, \underline{E}_0^* \cdot \underline{E}_{\mathsf{p}}, \underline{E}_0^* \cdot T \,\right\rangle$

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One *proof* = several *algorithms* for *computing* $\underline{E}_0^*$ or $\underline{E}_0^* \cdot \underline{E}_{\mathsf{p}}$

Automata and expressions

$$\mathsf{E}_2 = (a^* + b^*)^*$$

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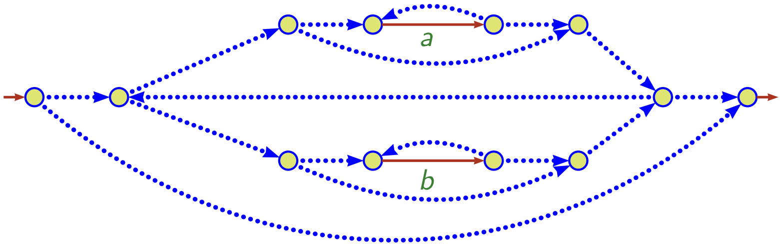

The Thompson automaton of E_2

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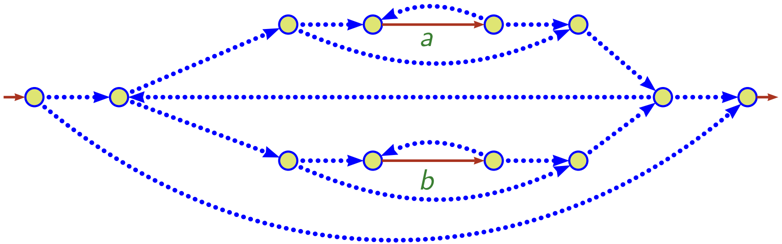

The Thompson automaton of E_2

Theorem (Folk–Lore ?)

The closure of the Thompson automaton of E

yields the Glushkov automaton of E

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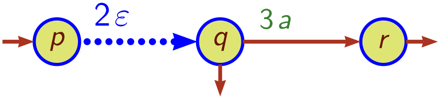

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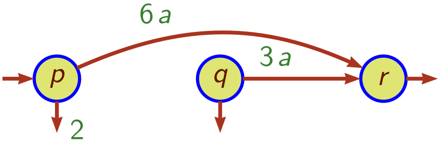

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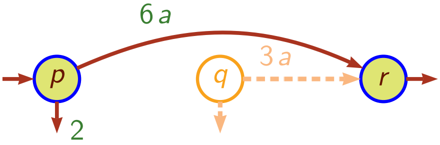

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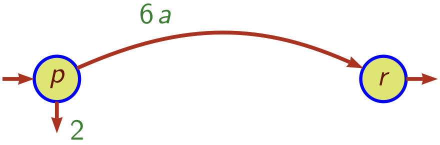

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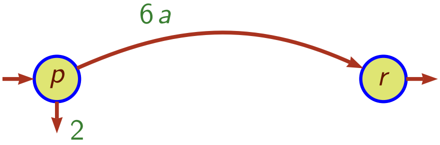

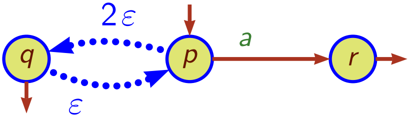

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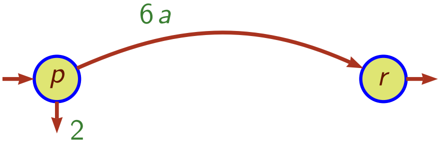

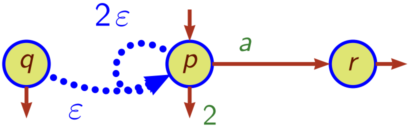

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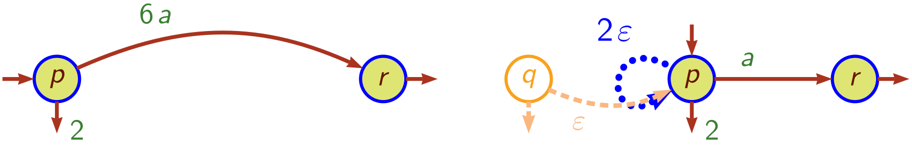

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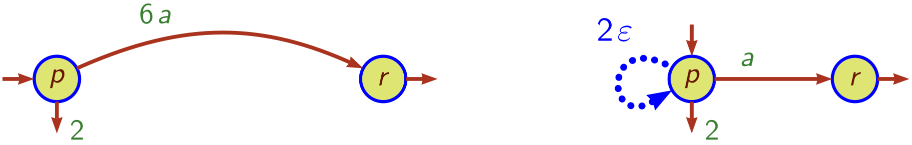

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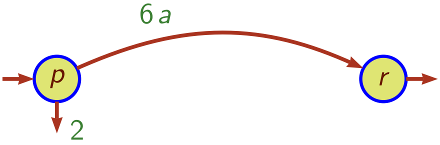

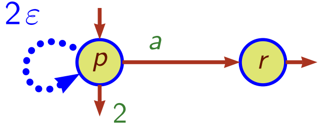

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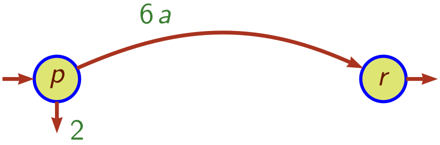

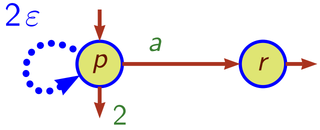

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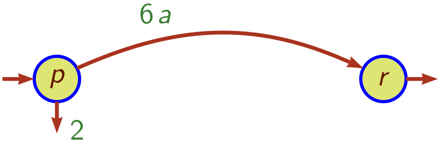

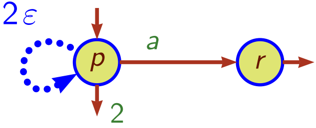

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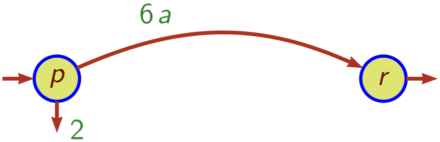

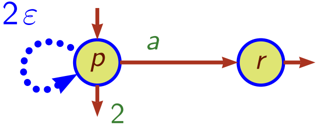

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$$a \quad \longmapsto \quad 1 + 2 + 4 + \cdots$$

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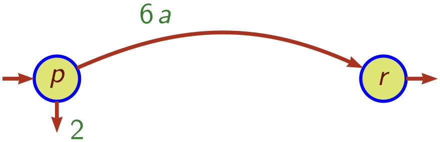

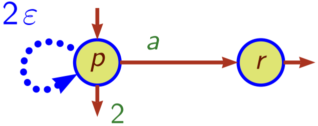

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$$a \quad \longmapsto \quad 1 + 2 + 4 + \cdots$$

undefined

A basic question in weighted automata theory

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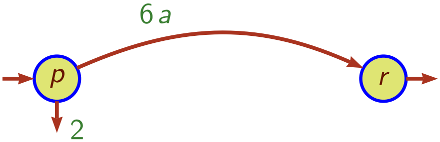

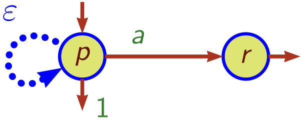

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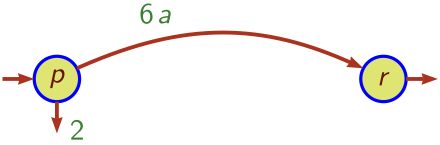

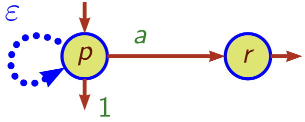

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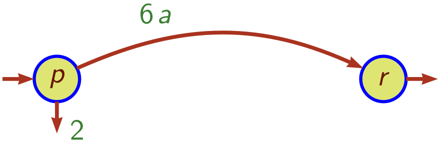

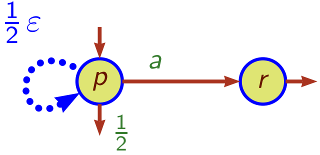

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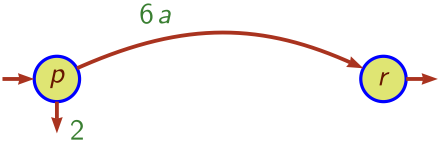

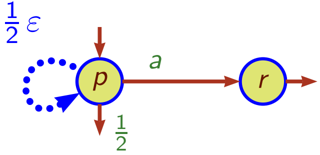

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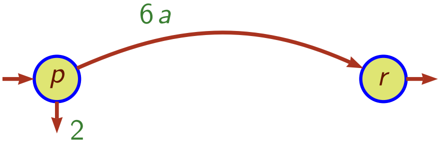

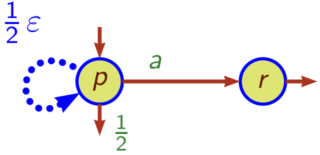

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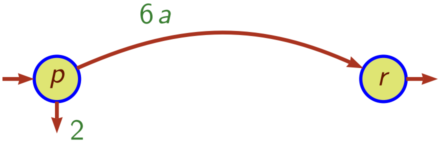

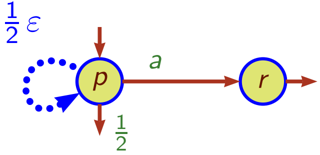

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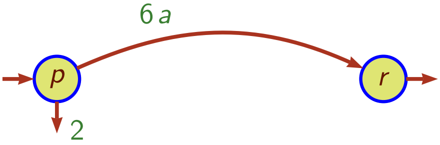

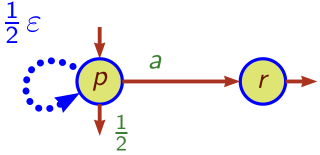

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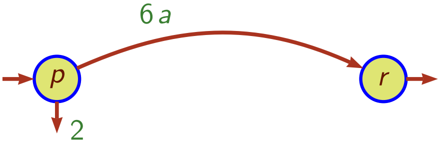

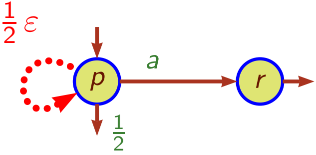

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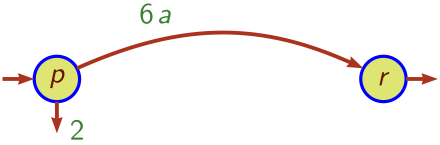

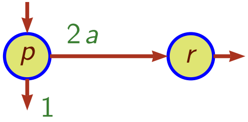

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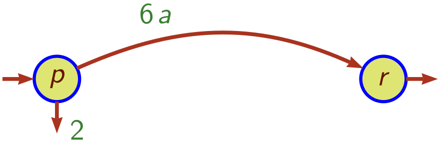

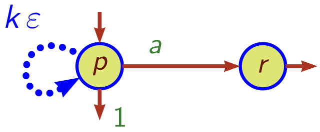

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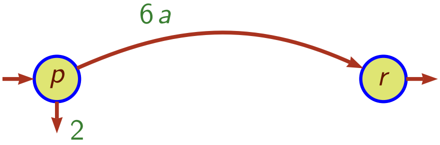

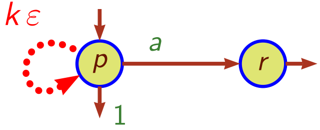

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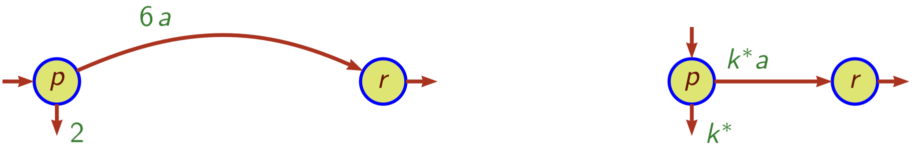

if $k^* = \sum_{n=0}^{\infty} k^n$ is defined in $\mathbb{K}$

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How to decide if the behaviour of an ε-WFA is *well-defined*?

Behaviour of weighted automata

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acyclic $\mathbb{K}$-automata

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First solution

behaviour well-defined $\Longleftrightarrow$ acyclic

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Legitimate, as far as the behaviours of the automata are concerned

(Kuich–Salomaa 86, Berstel–Reutenauer 84-88; 11)

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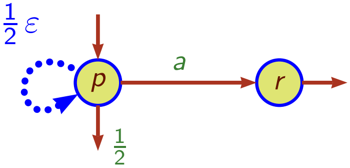

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First point of view (algebraico-logic)

- Definition of a *new operator for infinite sums* $\sum_I$
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Works of Bloom, Ésik, Kuich (90's –)

based on the axiomatisation described by Conway (72)

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Infinite sums are given a meaning via a topology on $\mathbb{K}$

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Topology allows to define summable families in $\mathbb{K}$

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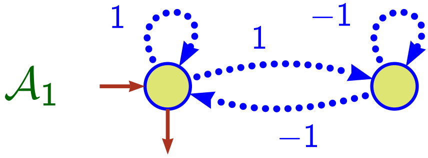
- Yields a consistent theory
- Two pitfalls for effectivity
  - *effective computation* of a summable family may not be possible
  - *effective computation* may give values to non summable families

Problems in computing the behaviour of a weighted automaton



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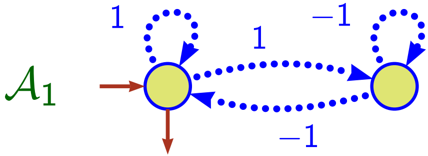

$$\mathcal{A}_1 = \langle\, I_1, \underline{E_1}, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

$$|\mathcal{A}_1| = I_1 \cdot \underline{E_1}^* \cdot T_1$$

$$\underline{E_1}^2 = 0 \quad \Longrightarrow \quad \underline{E_1}^* = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow \quad |\mathcal{A}_1| = 2$$

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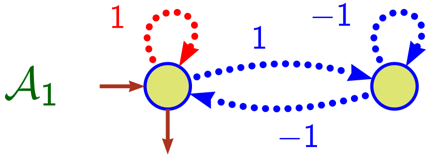

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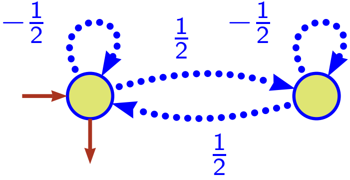
$$\underline{E_1}^{2} = 0 \quad \Longrightarrow \quad \underline{E_1}^{*} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow \quad |\mathcal{A}_1| = 2$$

Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_2$



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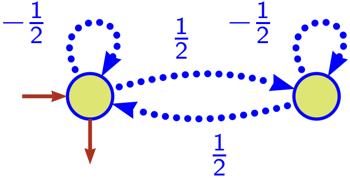

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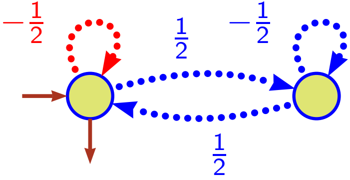

$$\left(-\frac{1}{2}\right)^* = \frac{2}{3}$$

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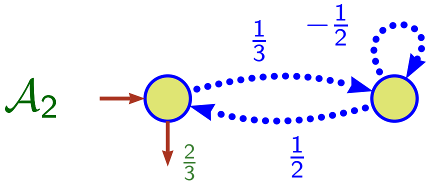

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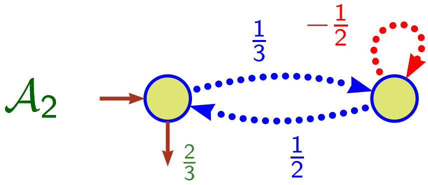

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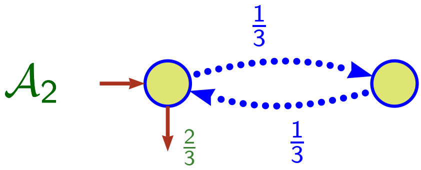

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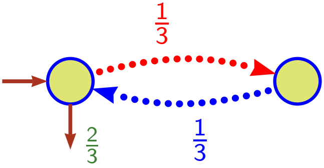

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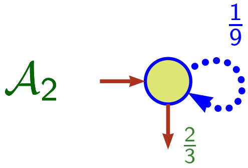

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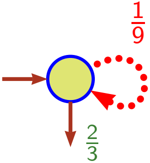
Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_2$



$$\left(\frac{1}{9}\right)^* = \frac{9}{8}$$

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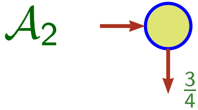

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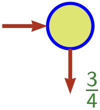

$$|\mathcal{A}_2| = \frac{3}{4}$$

$$\mathcal{A}_2 = \langle\, I_2, \underline{E_2}, T_2 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

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A chicken and egg problem

automaton $\mathcal{A}$

algorithm A

A chicken and egg problem

automaton	**algorithm**
$\mathcal{A}$	A
valid ?	**success ?**

A chicken and egg problem

automaton		**algorithm**
$\mathcal{A}$		A
valid ?		**success ?**
valid	$\Longrightarrow$	**success**

A chicken and egg problem

automaton		algorithm
$\mathcal{A}$		A
valid ?		success ?
valid	$\Longrightarrow$	success
valid	$\overset{?}{\Longleftarrow}$	success

A new definition of validity for weighted automata

$\mathcal{A} = \langle\, \mathbb{K}, A, Q, I, E, T\, \rangle$ possibly with ε-transitions

E^* *free monoid* generated by E

$\mathsf{P}_{\mathcal{A}}$ *set of paths* in $\mathcal{A}$ (local) rational subset of E^*

Definition
R rational family of paths of $\mathcal{A}$ $\quad R \in \operatorname{Rat} E^* \ \wedge\ R \subseteq \mathsf{P}_{\mathcal{A}}$

Definition
$\mathcal{A}$ is valid iff
$\forall R$ rational family of paths of $\mathcal{A}$, $\mathbf{WL}(R)$ is summable

A new definition of validity for weighted automata

Validity implies the well-definition of behaviour

The notion of validity settles the previous examples

Remark

If every subfamily of a summable family in $\mathbb{K}$ is summable,
then validity is equivalent to the well-definition of behaviour

Eg. $\mathbb{R}$, $\mathbb{C}$ *(and* $\mathbb{N}$, $\mathbb{Z}$, $\mathcal{N}$ *).*

If every rational subfamily of a summable family in $\mathbb{K}$ is summable,
then validity is equivalent to the well-definition of behaviour

Eg. $\mathbb{Q}$.

A new definition of validity for weighted automata

Theorem

$\mathcal{A}$ is valid iff the behaviour of every covering of $\mathcal{A}$ is well-defined

Theorem

If $\mathcal{A}$ is valid, then 'every' removal algorithm on $\mathcal{A}$ is successful

Nota Bene

We do not know yet how to decide whether a $\mathbb{Q}$ - or an $\mathbb{R}$ -automaton is valid.

Deciding validity

Straightforward cases

- Non starable semirings (eg. $\mathbb{N}$, $\mathbb{Z}$)

  $\mathcal{A}$ valid $\iff$ $\mathcal{A}$ acyclic
- Complete topological semirings (eg. $\mathcal{N}$) every $\mathcal{A}$ valid
- Rationally additive semirings (eg. $\mathrm{Rat}\, A^*$) every $\mathcal{A}$ valid
- Locally closed commutative semirings every $\mathcal{A}$ valid

Deciding validity

Definition
$\mathbb{K}$ topological, ordered, positive, star-domain downward closed
(TOP SDDC)

$\mathbb{N}$, $\mathcal{N}$, $\mathbb{Q}_+$, $\mathbb{R}_+$, $\mathbb{Z}$min, $\mathrm{Rat}\,A^*$,... are TOP SDDC

$\mathbb{N}_\infty$, (binary) positive decimals,... are not TOP SDDC

Theorem
$\mathbb{K}$ topological, ordered, positive, star-domain downward closed
A $\mathbb{K}$-automaton is valid if and only if
the ε-removal algorithm succeeds

Deciding validity

Definition

If $\mathcal{A}$ is a $\mathbb{Q}$- or $\mathbb{R}$-automaton,

then $\mathsf{abs}(\mathcal{A})$ is a $\mathbb{Q}_+$- or $\mathbb{R}_+$-automaton

Theorem

A $\mathbb{Q}$- or $\mathbb{R}$-automaton $\mathcal{A}$ is valid if and only if $\mathsf{abs}(\mathcal{A})$ is valid.

Hidden parts

- The removal algorithm itself
- Details on the topology we put on semirings
- Automata and expressions validity
- Validity of automata and covering
- ‘Infinitary’ axioms : *strong*, *star-strong* semirings
- Links with the ‘axiomatic’ approach (Bloom–Ésik–Kuich)
- References to previous work (on removal algorithms):

Conclusion

- Semiring structure is weak, topology does not help so much.
- This weakness imposes a restricted definition of validity,
  in order to guarantee success of validity algorithms.
- Axiomatic approach does not allow to deal
  with most common numerical semirings such as $\mathbb{Z}$min, or $\mathbb{Q}$.
- On 'usual' semirings,
  the new definition of validity coincides with the former one.

Conclusion (2)

- Apart the trivial cases, and the TOP SDDC case,
  decision of validity is never granted, and is to be established.
- On ‘usual’ semirings, validity is decidable.
- The new definition of validity
  fills the ‘effectivity gap’ left open by the former one.
- The algorithms implemented in AWALI
  are given a theoretical framework.

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All's well, that ends well!

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  - Termination issues (weighted versus Boolean cases)
  - Complexity issues

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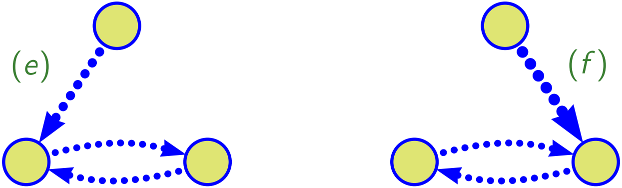
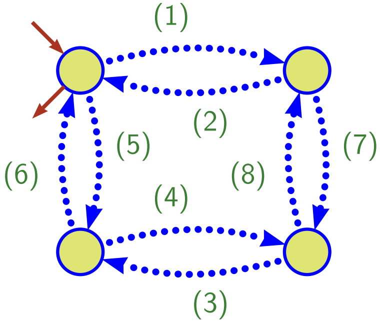


Boolean ε-removal procedure does not terminate

if newly created ε-transitions are stored in a stack

Hidden parts

- The removal algorithm itself
  - Termination issues (weighted versus Boolean cases)
  - Complexity issues



weighted ε-removal procedure does not terminate

if newly created ε-transitions are stored in a queue

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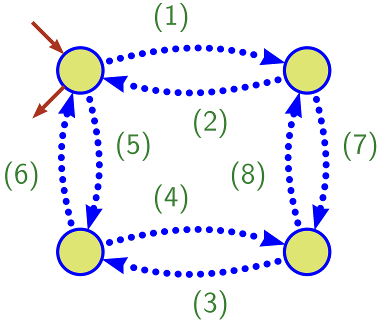
  - Termination issues (weighted versus Boolean cases)
  - Complexity issues



A *state-elimination-like* algorithm insures

termination of ε-removal procedure

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- Details on the topology we put on semirings

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Definition

$\mathbb{K}$ *topological*: $\mathbb{K}$ regular Hausdorff $\oplus, \otimes$ continuous

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Definition

$\{t_i\}_{i \in I}$ *summable* of sum t :

$$\forall V \in \mathrm{N}(t)\,,\ \exists J_V \text{ finite}\,,\ J_V \subset I\,,\ \forall L \text{ finite}\,,\ J_V \subseteq L \subset I \quad \sum_{i \in L} t_i \in V\,.$$

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Lemma (Associativity)

$\{t_i\}_{i \in I}$ *summable of sum* t ,
$I = \bigcup_{j \in J} K_j \quad \forall j \in J \quad \{t_i\}_{i \in K_j}$ *summable of sum* s_j ,
then $\{s_j\}_{j \in J}$ *summable of sum* t

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Automata and expressions validity

'Kleene' theorem

Automata	$\Longleftrightarrow$	Expressions
$\mathcal{A}$	$\Longleftrightarrow$	E
Weighted automata	$\Longleftrightarrow$	Weighted expressions

Automata and expressions validity

'Kleene' theorem

Automata	$\Longleftrightarrow$	Expressions
$\mathcal{A}$	$\Longleftrightarrow$	E
Weighted automata	$\Longleftrightarrow$	Weighted expressions

Validity of expressions

E *valid* $\Longleftrightarrow$ c(E) well-defined

c(E) computed by a bottom-up traversal of the syntactic tree of E

Automata and expressions validity

Valid $\mathcal{A}$ yields valid E

Valid E yields valid $\mathcal{A}$ with Glushkov construction

Valid E may yield non valid $\mathcal{A}$ with Thompson construction

Automata and expressions validity

Valid $\mathcal{A}$ yields valid E

Valid E yields valid $\mathcal{A}$ with Glushkov construction

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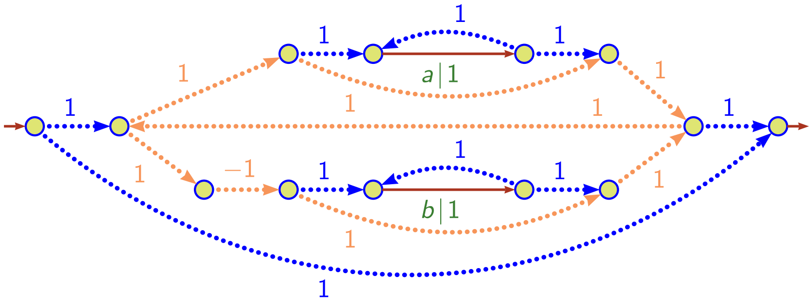

The Thompson automaton of $(a^* + \{-1\}b^*)^*$

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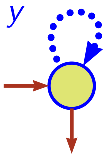
Validity of automata and covering

$\mathcal{A}_5$



$$\mathbb{S} \subset \mathbb{N}^{2\times 2}, \quad x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{\mathbb{S}}, \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x+y = \infty_{\mathbb{S}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\mathbb{S}$ equipped with the discrete topology

$0_{\mathbb{S}}$, y, and $\infty_{\mathbb{S}}$ starable $\qquad x = y^2 \qquad x$ not starable

Validity of automata and covering

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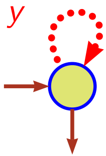

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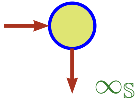

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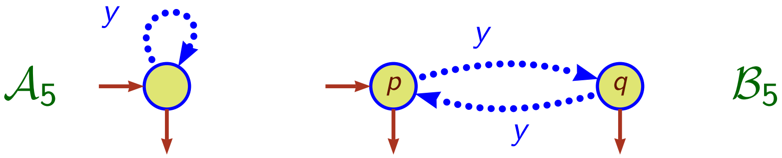

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Validity of automata and covering

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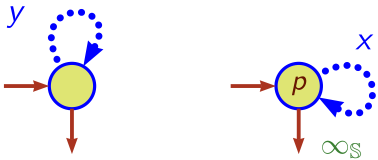

$\mathcal{B}_5$

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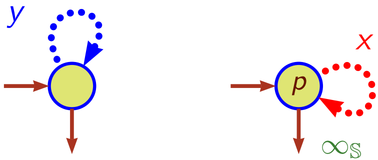

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Definition

A topological semiring is a *strong* semiring
if the product of two summable families is a summable family

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- ‘Infinitary’ axioms : *strong*, *star-strong* semirings

Definition

A topological semiring is a *strong* semiring
if the product of two summable families is a summable family

Theorem

$\mathbb{K}$ *strong semiring* $\quad s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ *starable iff* $s_0 \in \mathbb{K}$ *starable*

Hidden parts

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Proposition (Madore 18)

There exist (semi)rings $\mathbb{K}$ *that are not strong*

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A topological semiring is a *star-strong* semiring if
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A strong semiring $\mathbb{K}$ is starable and star-strong iff every rational family of $\mathbb{K}$ is summable

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Conjecture

A starable strong semiring star-strong

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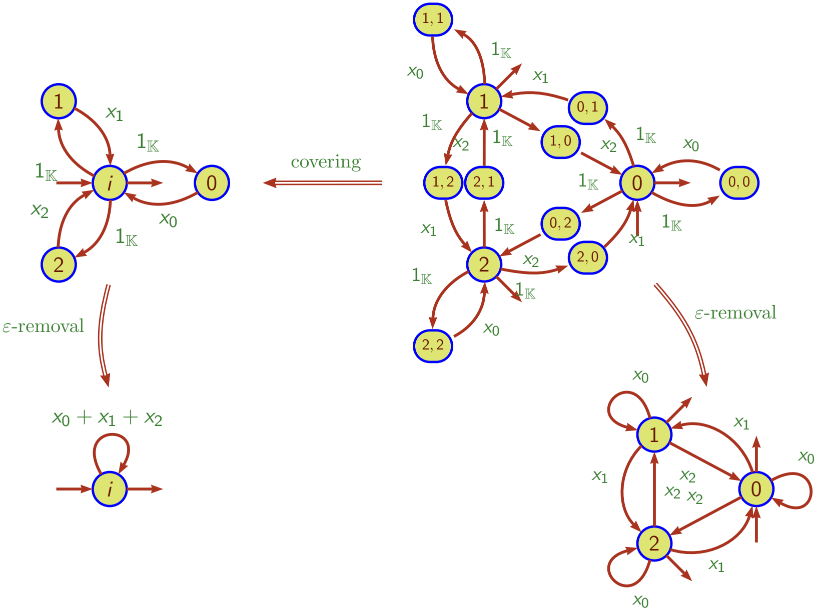
Theorem

A starable star-strong semiring is an iteration semiring

Group identities



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- References to previous work (on removal algorithms)
  - *locally closed* srgs (Ésik–Kuich), *k-closed* srgs (Mohri)
  - links with other algorithms:
    shortest-distance algorithm (Mohri),
    state-elimination method (Hanneforth–Higueira)